FOUCAULT'S PENDULUM IN ELLIPTIC SPACE*

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1. In the following for e- read euclidean, for E- read elliptic. Let x, y, z be ordinary rectangular coördinates of a point in e-space whose origin is O. Set $r^2=x^2+y^2+z^2$, $\lambda=4R^2+r^2$, $\mu=4R^2-r^2$, where R is an arbitrary positive constant. For all points of e-space

$$d\sigma^2 = dx^2 + dy^2 + dz^2.$$

For points within and on the e-sphere $\mu=0$ we establish an elliptic metric by means of

$$ds = (4R^2/\lambda)d\sigma. \tag{1}$$

Points outside of $\mu=0$ do not exist in E-space while two diametral points on $\mu=0$ are regarded as identical.

An E-straight is an e-circle cutting $\mu=0$ in diametral points; an E-plane is an e-sphere cutting $\mu=0$ along a great circle. The e-sphere $\mu=0$ is regarded as an E-plane. Angles between E-straights and planes have the same measure in E- as in e-space.

The 4 E-planes $x=0$, $y=0$, $z=0$, $\mu=0$ form an E-tetrahedron which we call τ. From a point xyz drop E-perpendiculars on the 4 faces of τ and let δ_i, $i=1, 2, 3, 4$, be their E-lengths. We set

$$z_i = R\sin(\delta_i/R).$$

We find

$$z_1 = 4R^2x/\lambda,\quad z_2 = 4R^2y/\lambda,\quad z_3 = 4R^2z/\lambda,\quad z_4 = R\mu/\lambda.$$

Also

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = R^2,\qquad ds^2 = dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2. \tag{2}$$

In these coördinates the equation of an E-plane has the form

$$a_1z_1 + a_2z_2 + a_3z_3 + a_4z_4 = 0.$$

The distance δ between two points z, z' is given by

$$\cos(\delta/R) = \frac{z_1z_1' + z_2z_2' + z_3z_3' + z_4z_4'}{R^2}.$$

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We may without loss of generality set $R=1$ and this will be done in the following.

2. Let $c_1, \cdots, c_4$ be the coördinates of the point of suspension O' whose latitude is ϕ and whose longitude is θ. Let $\overline{OO'}=\rho$ in E-measure. For brevity we set

$$r = \sin\rho, \quad r' = \cos\rho, \quad r_1 = \cot\rho\,; \quad p = \sin\phi, \quad p' = \cos\phi.$$

Then

$$c_1 = rp'\cos\theta, \quad c_2 = rp'\sin\theta, \quad c_3 = rp, \quad c_4 = r'.$$

Let us now displace the xyz axes so that O moves to O'. The new e-axes call ξ, η, ζ, where $+\xi$, $+\eta$ point south and east respectively, while $+\zeta$ points to the zenith. These axes define a new E-tetrahedron which we call τ'.

The relation between the coördinates $z_1, \cdots, z_4$ referred to τ and the coördinates $\zeta_1, \cdots, \zeta_4$ of the same point referred to τ' is given by the table, read as in ordinary analytic geometry.

(3)

	z_1	z_2	z_3	z_4
ζ_1	$p\cos\theta$	$p\sin\theta$	$-p'$	0
ζ_2	$-\sin\theta$	$\cos\theta$	0	0
ζ_3	$r'p'\cos\theta$	$r'p'\sin\theta$	$r'p$	$-r$
ζ_4	$rp'\cos\theta$	$rp'\sin\theta$	rp	r'

We now suppose that τ remains fixed in space, that the earth rotates about the z axis with a constant angular velocity $k=\dot\theta=d\theta/dt$ and that finally τ' is rigidly attached to the earth.

We suppose the bob B of the pendulum to be a particle of mass m, and attached to the point of suspension c or O' by a weightless rod of length L in E-measure. Set $l=\sin L$, $l'=\cos L$; let the plane through B and the ζ axis make the angle ω with the $\xi\cdot\zeta$ plane, let the rod $O'B$ make with the negative ζ axis the angle ψ. Then the coördinates of B relative to τ' are

$$\zeta_1 = l\sin\psi\cos\omega, \quad \zeta_2 = l\sin\psi\sin\omega, \quad \zeta_3 = -l\cos\psi, \quad \zeta_4 = l'. \tag{4}$$

3. Let the force F act on a particle; if the particle is displaced along an elementary segment of length ds as defined by (1) or by (2) and if θ is the angle between F and ds we assume with Killing* that the work done is $dW=F\cos\theta ds$. We ask now what is dW when ψ receives the increment $d\psi$. In the triangle $OO'B$ we have setting $\overline{OB}=\beta$ in E-measure

* W. Killing, *Die Mechanik in den nicht-euklidischen Raumformen*, Crelle's Journal, vol. 98 (1885), p. 1.

$$\sin B = \frac{\sin \rho}{\sin \beta}; \ \sin \psi = -\cos \theta.$$

As $ds = \sin L d\psi$ we have

$$dW = -F \frac{\sin \rho}{\sin \beta} \sin L \sin \psi d\psi = -F \frac{\sin \rho}{\sin \beta} d\zeta_3. \tag{5}$$

Since the length of the pendulum L is negligible compared with ρ, $\sin \beta = \sin \rho$ with a high degree of exactitude. We may therefore write

$$dW = -F d\zeta_3 = -F \sin L \sin \psi d\psi, \tag{6}$$

which is what we would expect at once.

We note that the work done when ω receives an increment is 0, since in this case $\theta = \pi/2$, hence $\partial W/\partial \omega = 0$.

4. We now wish to calculate the velocity v of the bob B. We have

$$v^2 = \dot{s}^2 = \dot{z}_1^2 + \dot{z}_2^2 + \dot{z}_3^2 + \dot{z}_4^2.$$

From the table (3) we express the z's in terms of the ζ's and these by means of (4) in terms of ψ, ω. We then differentiate the z's, squared, and add. We find, setting as before $k = \dot{\theta}$,

$$\begin{aligned} v^2 = {} & k^2\left[l^2 \sin^2 \psi \sin^2 \omega + (pl \sin \psi \cos \omega - r'p'l \cos \psi + l'p'r)^2\right] \\ & + l^2\dot{\psi}^2 + l^2 \sin^2 \psi \dot{\omega}^2 \\ & + 2k\dot{\psi}\left[rp'll' \cos \psi \sin \omega - l^2 r'p' \sin \omega\right] \\ & + 2k\dot{\omega}\left[l^2 p \sin^2 \psi - r'p'l^2 \sin \psi \cos \psi \cos \omega + rp'll' \sin \psi \cos \omega\right]. \end{aligned} \tag{7}$$

The kinetic energy of the bob B we define by

$$T = \tfrac{1}{2} m v^2.$$

5. We assume now that the motion of the bob B takes place according to Hamilton's principle

$$\int (\delta T + \delta W) dt = 0.$$

On performing the variation we get as usual Lagrange's equation

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\omega}} - \frac{\partial T}{\partial \omega} = 0, \qquad \frac{d}{dt} \frac{\partial T}{\partial \dot{\psi}} - \frac{\partial T}{\partial \psi} = \frac{\partial W}{\partial \psi}. \tag{8}$$

Let us calculate the ω equation. From (7)

$$\frac{\partial T}{\partial \dot{\omega}} = l^2(\sin^2 \psi) \cdot \dot{\omega} + k\left[l^2 p \sin^2 \psi - r'p'l^2 \sin \psi \cos \psi \cos \omega + rp'll' \sin \psi \cos \omega\right],$$

$$\frac{\partial T}{\partial \omega} = k^2[l^2 \sin^2 \psi \sin \omega \cos \omega - pl \sin \psi \sin \omega(pl \sin \psi \cos \omega - r'p'l \cos \psi + l'p'r)]$$
$$+ k\dot{\psi}[rp'll' \cos \psi \cos \omega - l^2 r'p' \cos \omega]$$
$$+ k\dot{\omega}[r'p'l^2 \sin \psi \cos \psi \sin \omega - rp'll' \sin \psi \sin \omega].$$

Thus the first equation (8) gives

$$l^2\frac{d}{dt}(\dot{\omega} \sin^2 \psi) + kl^2\frac{d}{dt}(\sin^2 \psi) - kr'p'\frac{l^2}{2}\,\frac{d}{dt}(\sin 2\psi \cos \omega) + krp'll'\frac{d}{dt}(\sin \psi \cos \omega)$$

$$\begin{aligned}(9)\qquad &= k^2 l^2 \sin^2 \psi \sin \omega \cos \omega - k^2 pl \sin \psi \sin \omega(pl \sin \psi \cos \omega - r'p'l \cos \psi + l'p'r) \\ &\quad + krp'll'((\cos \omega \cos \psi)\dot{\psi} - (\sin \psi \sin \omega)\dot{\omega}) \\ &\quad - kl^2 r'p'((\cos \omega)\dot{\psi} - (\sin \psi \cos \psi \sin \omega)\dot{\omega}).\end{aligned}$$

We will now suppose that ψ is so small that we may set $\sin \psi = \psi$ without sensible error; then (9) becomes

$$l^2 \frac{d}{dt}(\dot{\omega}\psi^2) + kl^2 p \frac{d}{dt}(\psi^2) - kr'p'l^2 \frac{d}{dt}(\psi \cos \omega) + krp'll'\frac{d}{dt}(\psi \cos \omega)$$
$$= k^2 l^2 \psi^2 \sin \omega \cos \omega - k^2 pl\psi \sin \omega\{pl\psi \cos \omega + (l'r - r'l)p'\}$$
$$+ krp'll'\frac{d}{dt}(\psi \cos \omega) - kl^2 r'p'\frac{d}{dt}(\psi \cos \omega) ;$$

or as $l'r - r'l = \cos L \sin \rho - \cos \rho \sin L = \sin (\rho - L) = \sin \rho = r$ very nearly, we get

$$l(\ddot{\omega}\psi^2 + 2\psi\dot{\omega}\dot{\psi} + 2kp\psi\dot{\psi})$$
$$= k^2 l\psi^2 \sin \omega \cos \omega - k^2 p^2 l\psi^2 \sin \omega \cos \omega - k^2 pp'r\psi \sin \omega.$$

Hence

$$2\dot{\psi}(\dot{\omega} + kp) + \psi\ddot{\omega} = k^2 p'^2 \psi \sin \omega \cos \omega - (k^2 pp'r/l)\sin \omega.$$

These are entirely analogous to the equations of classical mechanics. Under similar conditions we may say therefore that in first approximation the angular velocity of the plane of vibration is

$$\dot{\omega} = - k \sin \phi.$$

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